MATH 121, Calculus I — Final Exam (Spring 2013)

May 15, 2013 — 4:30pm to 7:00pm

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Lab Instructor: ____________________

The exam has a total value of 330 points that includes 300 points for the regular exam problems and 30 points for the extra credit problem (Problem number 23). The exam contains two distinct parts. Part I contains 18 multiple-choice problems with each problem worth 10 points. Part II contains 5 show-your-work problems with each problem worth 30 points. The exam contains a total of 23 problems. The exam is strictly closed-book and closed-notes. THE USE OF CALCULATORS IS NOT ALLOWED.

Score

Problem 1		Problem 13	
Problem 2		Problem 14	
Problem 3		Problem 15	
Problem 4		Problem 16	
Problem 5		Problem 17	
Problem 6		Problem 18	
Problem 7		Problem 19	
Problem 8		Problem 20	
Problem 9		Problem 21	
Problem 10		Problem 22	
Problem 11		Problem 23	
Problem 12		Total score	

Part I — Multiple-Choice Problems

Instructions: Write the letter corresponding to each of your answers in the blank box that is provided. Correct answers do not require work to receive full credit. However, partial credit can be awarded for incorrect answers based on the work that is shown in the adjacent blank spaces. Hence, you are strongly advised to show your work for each problem.

(1) [**10 points**] Determine which of the following is an equation of the tangent line to the curve $y = \sqrt{x}$ at the point $(9, 3)$.

(A) $y = 6x - 51$.

(B) $y = 3x + 24$.

(C) $y = \frac{1}{6}x + \frac{3}{2}$.

(D) $y = \frac{\sqrt{x}}{2} - \frac{9}{2\sqrt{x}} + 3$.

Answer: C

$$y' = \frac{1}{2\sqrt{x}}$$

$$y'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

$$\Rightarrow \quad y - 3 = \frac{1}{6}(x-9)$$

$$y = 3 + \frac{1}{6}x - \frac{9}{6}$$

$$\boxed{y = \frac{1}{6}x + \frac{3}{2}}$$

(2) [**10 points**] If $x^2y + xy^2 = 3x$, then $\frac{dy}{dx}$ is

(A) $\frac{x2 + xy^2}{3}$.

(B) $\frac{3 - 2xy - y^2}{x^2 + 2xy}$.

(C) $2x^2y + y^2$.

(D) $\frac{2x + 3}{x^2 + x}$.

Answer: B

$$2xy + x^2 \cdot \frac{dy}{dx} + y^2 + x \cdot 2y \cdot \frac{dy}{dx} = 3$$

$$(x^2 + 2xy)\frac{dy}{dx} = 3 - 2xy - y^2$$

$$\Rightarrow \quad \boxed{\frac{dy}{dx} = \frac{3 - 2xy - y^2}{x^2 + 2xy}}$$

(3) [**10 points**] $F(x) = \int_0^x sin(t)dt$ for $0 \leq x \leq 2\pi$. F is increasing only in the open interval(s)

(A) $(\frac{\pi}{2}, \pi)$.

(B) $(0, \frac{\pi}{4}), (\frac{5\pi}{4}, 2\pi)$.

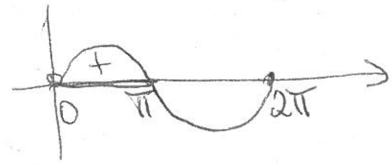
(C) $(0, \pi)$.

(D) $(\frac{3\pi}{4}, \pi)$.

Answer: $\boxed{C}$

$F'(x) = \sin x$

$F'(x) > 0$ iff $\sin x > 0$



iff $0 < x < \pi$

(4) [**10 points**] Evaluate $\lim_{x \to 0} \frac{1 - \cos(4x)}{x^2}$.

(A) 8.

(B) 4.

(C) 2.

(D) 1.

Answer: $\boxed{A}$

$$\overset{L'H}{=} \lim_{x \to 0} \frac{4 \sin 4x}{2x} \overset{L'H}{=} \lim_{x \to 0} \frac{16 \cos 4x}{2}$$

$$= \frac{16 \cdot 1}{2} = 8$$

(5) **[10 points]** Evaluate $\int_{-2}^{1} |x|\, dx$.

(A) $-\frac{5}{2}$.

(B) $-\frac{3}{2}$.

(C) $\frac{3}{2}$.

(D) $\frac{5}{2}$.

Answer: D

Since $|x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$

then $\int_{-2}^{1} |x|\,dx = \int_{-2}^{0} (-x)\,dx + \int_{0}^{1} x\,dx$

$= \left[\frac{-x^2}{2}\right]_{-2}^{0} + \left[\frac{x^2}{2}\right]_{0}^{1}$

$= 0 + 2 + \frac{1}{2} - 0 = \frac{5}{2}$

(6) **[10 points]** Find the largest open interval on which $f(x) = xe^x$ is concave upward. $f''(x) > 0$

(A) $(0, \infty)$.

(B) $(-1, \infty)$.

(C) $(-2, \infty)$.

(D) $(-\infty, \infty)$.

Answer: C

$f'(x) = e^x + xe^x = e^x(1+x)$

$f''(x) = e^x(1+x) + e^x = e^x(2+x)$

$f''(x) > 0$ iff $2 + x > 0$ iff $\boxed{x > -2}$

(7) [**10 points**] Determine which of the following equals $\int x\sqrt{x^2+1}\,dx$.

(A) $\frac{1}{3}x^2(x^2+1)^{3/2}+c$.

(B) $\frac{1}{3}(x^2+1)^{3/2}+c$.

(C) $\frac{1}{2}x^2(x^2+1)^{3/2}+c$.

(D) $\frac{1}{2}(x^2+1)^{3/2}+c$.

Answer: $\boxed{B}$

let $x^2+1=u$

$2x\,dx = du$

$\Rightarrow x\,dx = \frac{1}{2}du$

$\Rightarrow \int x\sqrt{x^2+1}\,dx$

$= \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{2}\int u^{1/2}\,du$

$= \frac{1}{2}\frac{u^{3/2}}{3/2} = \frac{1}{3}(x^2+1)^{3/2}+C$

(8) [**10 points**] Let f and g be differentiable functions defined on $(-\infty,\infty)$. Suppose we have the following table of values for f, g, f' and g'.

x	0	1	2	3	4	5
$f(x)$	-7	30	4	-10	8	6
$g(x)$	14	7	-11	2	36	12
$f'(x)$	1	-4	5	0	30	8
$g'(x)$	18	-3	24	4	6	-2

Using this table, find the value of $(f\circ g)'(3)$. $= f'(g(3))\cdot g'(3)$

$= f'(2)\cdot 4 = 20$

(A) -20.

(B) 0.

(C) 20.

(D) 32.

Answer: $\boxed{C}$

(9) **[10 points]** Determine which of the following equals $\int x^2 \ln(x)\, dx$.

(A) $\frac{1}{3}x^3 \ln(x) - \frac{1}{9}x^3 + c.$

(B) $\frac{1}{3}x^3 \ln(x) - \frac{1}{6}x^3 + c.$

(C) $\frac{1}{3}x^3 \ln(x) - \frac{1}{9}x^2 + c.$

(D) $\frac{1}{3}x^3 \ln(x) - \frac{1}{6}x^2 + c.$

Answer: A

let $u = \ln x \qquad dv = x^2 dx$

$du = \frac{1}{x} dx \qquad v = \frac{x^3}{3}$

$\Rightarrow = uv - \int v\, du$

$= \frac{x^3}{3}\ln x - \int \frac{x^3}{3} \cdot \frac{1}{x} dx$

$= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2 dx = \frac{x^3}{3}\ln x - \frac{1}{9}x^3 + C$

(10) **[10 points]** Determine the value of the definite integral $\int_0^4 \frac{1}{\sqrt{x}} dx$

(A) ∞.

(B) 4.

(C) 5.

(D) 3.

Answer: B

$\Rightarrow$ this is an improper integral

$= \lim_{t \to 0^+} \int_t^4 x^{-1/2} dx = \lim_{t \to 0^+} \left[\frac{x^{1/2}}{1/2} \right]_t^4$

$= \lim_{t \to 0^+} \left[2 \cdot 2 - 2\sqrt{t} \right] = 4$

(11) **[10 points]** If the volume V of a cube is decreasing at the rate of 24 in^3/sec, then find the rate at which the length of a side of the cube is decreasing when $V = 8\ \text{in}^3$.

(A) 1 in/sec.

(B) 2 in/sec.

(C) 3 in/sec.

(D) 4 in/sec.

Answer: B

$V = x^3$ where x is the length of side of a cube

$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$ if $V = 8$ then $V = x^3 = 8 \Rightarrow x = 2$

$24 = 3 \cdot 2^2 \cdot \frac{dx}{dt}$

$24 = 12 \frac{dx}{dt}$

$\Rightarrow \frac{dx}{dt} = 2$ in/sec

(12) **[10 points]** Determine which of the following definite integrals equals $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\sqrt{\frac{i}{n}}$.

(A) $\int_0^1 \sqrt{\frac{1}{x}}\,dx$.

(B) $\int_0^1 \sqrt{x}\,dx$.

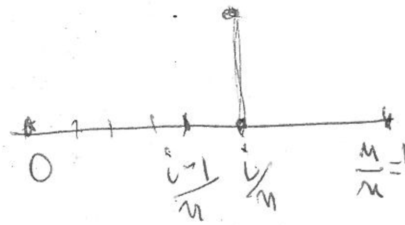
(C) $\int_0^1 4\sqrt{4x}\,dx$.

(D) $\int_0^1 4x\sqrt{4x}\,dx$.

Answer: B

$f(x) = \sqrt{x}$

$[a, b] = [0, 1] \Rightarrow \Delta x = \frac{1-0}{n} = \frac{1}{n}$



(13) **[10 points]** Let $s(t)$ be the displacement function of a mouse moving along the x-axis. Let $v(t)$ and $a(t)$ be its velocity and acceleration functions respectively. If

$$a(t) = 2 + 4e^{2t}, \quad v(0) = 1 \quad \text{and} \quad s(0) = 4,$$

determine which of the following expressions describes $s(t)$.

(A) $8e^{2t}$.

(B) $t^2 + e^{2t}$.

(C) $t^2 + 8e^{2t} - 3t - 4$.

(D) $t^2 + e^{2t} - t + 3$.

Answer: D

$v(t) = \int a(t)\,dt = \int (2 + 4e^{2t})\,dt$

$= 2t + 2e^{2t} + C_1$

$1 = v(0) = 2\cdot 0 + 2e^{2\cdot 0} + C_1$

$1 = 2 + C_1 \Rightarrow C_1 = -1$

$\Rightarrow v(t) = 2t + 2e^{2t} - 1$

$s(t) = \int v(t)\,dt$

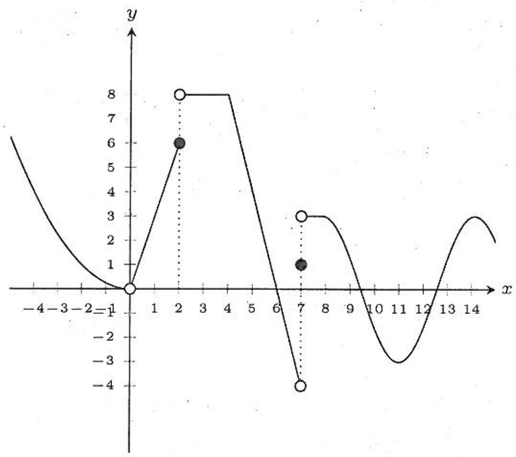
$= \int (2t + 2e^{2t} - 1)\,dt = t^2 + e^{2t} - t + C_2$

$4 = s(0) = 0^2 + e^{2\cdot 0} - 0 + C_2$

$4 = 1 + C_2 \Rightarrow C_2 = 3$

$\Rightarrow s(t) = t^2 + e^{2t} - t + 3$

(14) [**10 points**]The figure below shows part of the graph of a function f.



Using this figure, determine which of the following statements about f is false.

(A) $\lim_{x\to 0} f(x)$ exists.

(B) f is discontinuous at 2.

(C) It is continuous at 4.

(D) $\lim_{x\to 7} f(x)$ exists.

$\left.\begin{array}{l}\lim_{x\to 7^-} f(x) = -4 \\ \lim_{x\to 7^+} f(x) = 3\end{array}\right\} \Rightarrow \lim_{x\to 7} f(x)$ does not exist

Answer: $\boxed{\text{D}}$

(15) [**10 points**] Let f be a differentiable function defined on $(-\infty, \infty)$ whose derivative f' is continuous everywhere. Using the Fundamental Theorem of Calculus, determine which of the following equals $\int_x^{x^2} f'(t)\, dt$. $= f(x^2) - f(x)$

(A) $2x \cdot f(x^2) - f(x)$.

(B) $f(x^2)$.

(C) $f(x^2) - f(x)$.

(D) $f(x)$.

Answer: $\boxed{\text{C}}$

(16) **[10 points]** An inflection point of the function $f(x) = 2x^3 - 9x^2 - 24x - 10$ is

(A) 1.

(B) 1.5.

(C) 3.

(D) 4.

Answer: $\boxed{B}$

$f'(x) = 6x^2 - 18x - 24$

$f''(x) = 12x - 18$

$f''(x) = 0$ iff $12x - 18 = 0$

$x = \frac{18}{12}$

f'': $-$ $+$; $\frac{3}{2}$

f: $\cap$ $\cup$

$x = \frac{3}{2}$

$\left(\frac{3}{2}, f\left(\frac{3}{2}\right)\right)$ is the point of inflectio

(17) **[10 points]** Let a and b be two positive numbers. If $2a + 3b = 6$, then the maximum product of a and b is

(A) 1.5 .

(B) 2.

(C) 3.

(D) 3.5.

Answer: $\boxed{A}$

$$\max \rightarrow \begin{cases} P(a,b) = a \cdot b, & a, b \in \mathbb{R}^+ = (0, +\infty) \\ 2a + 3b = 6 \Rightarrow \end{cases}$$

$$a = \frac{6-3b}{2} > 0 \iff \underbrace{0 < b < 2}$$

$$P(b) = \frac{6-3b}{2} \cdot b$$

$$= \frac{6b - 3b^2}{2} = \frac{1}{2}(6b - 3b^2)$$

$$P'(b) = \frac{1}{2}(6 - 6b) = 0 \text{ iff } \underline{b = 1}$$

then $\underline{a = \frac{3}{2}}$

$$P''(b) = \frac{1}{2}(0 - 6) < 0 \iff$$

$$P(1) = -\frac{6}{2} < 0 \Longrightarrow \max$$

$$P_{max}\left(\frac{3}{2}, 1\right) = \frac{3}{2}$$

$P(0) = 0$ and $P(2) = 0$

$\Longrightarrow P_{max} = \frac{3}{2}$ is global max

(18) [**10 points**] Find the indefinite integral $\int \frac{3cos(ln(x))}{x} dx$ for $x > 0$.

(A) $3sin(ln(x)) + C$.

(B) $3cos(ln(x)) + C$.

(C) $3sec(ln(x)) + C$

(D) $3tan(ln(x)) + C$.

Answer: $\boxed{A}$

let $u = \ln x$

$du = \frac{1}{x} dx$

$= \int 3 \cos u \, du$

$= 3 \sin u = \underline{3 \sin(\ln x) + C}$

Part II — Show-Your-Work Problems

Instructions: Show all necessary work, and provide full justification for each answer. Circle your final answer(s).

(19) **[30 points]** If $f(x) = \dfrac{x^2 - 4x + 3}{x^2}$ then $f'(x) = \dfrac{4x - 6}{x^3}$ and $f''(x) = \dfrac{-8x + 18}{x^4}$.

$D_f = (-\infty, 0) \cup (0, +\infty)$

(a) Find the open intervals where f is increasing and where f is decreasing.

The critical points are: x=3/2 (because f'(x) = 0 <=> 4x-6=0 <=> x=3/2).
x=0 (because f'(0) is undefined).

Therefore: f is increasing on (- infinity, 0) U (3/2, + infinity) since f'(x) > 0 there.
f is decreasing on (0, 3/2) since f'(x) < 0 there.

(b) Find the open intervals where f is concave upward and where f is concave downward.

$f''(x) > 0$ iff $-8x + 18 > 0 \iff x < \frac{9}{4}$

concave up $\cup$ for $x \in (-\infty, 0) \cup (0, 9/4)$

$f''(x) < 0 \iff x > \frac{9}{4}$

concave down $\cap$ for $x \in (9/4, +\infty)$

(c) Find all local minima and local maxima for f if they exist.

f': $-$ $-$ $+$

0 $\quad$ $3/2$ critical point

f: $\searrow$ $\nearrow$

min

$$f_{min}\left(\frac{3}{2}\right) = \frac{\left(\frac{3}{2}\right)^2 - 4 \cdot \frac{3}{2} + 3}{\left(\frac{3}{2}\right)^2} = \frac{\frac{9}{4} - 6 + 3}{\frac{9}{4}} = 1 - \frac{3}{9/4} = 1 - \frac{12}{9} = 1 - \frac{4}{3} = -\frac{1}{3}$$

(20) [**30 points**] Find the value of the constant k for which the following piecewise-defined function is continuous everywhere. For the resulting function determine where the function is not differentiable. Justify your answers.

$$f'(x)=\begin{cases} 7 & \text{if } x<2 \\ -4 & \text{if } x>2 \end{cases}$$

$f'_- \neq f'_+$

$$f(x) = \begin{cases} 7x + k & \text{if } x \le 2; \\ 18 + kx & \text{if } x > 2. \end{cases} = \begin{cases} 7x-4 & \text{if } x \le 2 \\ 18-4x & \text{if } x>2 \end{cases}$$

functions $y_1 = 7x+k$ and $y_2 = 18+kx$ are continuous in their domains as polynomials of the first degree

the only point that we need to check continuity is $x=2$

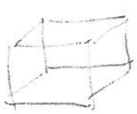
$f(2) = 14+k$

$\lim_{x\to 2^-} (7x+k) = 14+k$

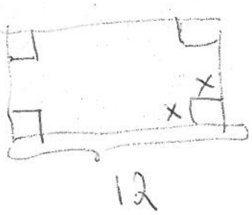
$\lim_{x\to 2^+} (18+kx) = 18+2k$

$\Rightarrow 14+k = 18+2k$, $k=-4$ $\Rightarrow f(x)=\begin{cases} 7x-4 & \text{for } x\le 2 \\ 18-4x & \text{for } x>2 \end{cases}$ is cont. but not diff. bc $f'_- \neq f'_+$

(21) [**30 points**] An open top box is to be made by cutting small identical squares from the corners of a 12 × 12 inch sheet of tin and bending up the remaining sides. How large should the squares cut from the corners be to make the box hold as much as possible (maximum volume)?



max $\Rightarrow V = (12-2x)^2 \cdot x$, $\boxed{0<x<6}$



$V(x) = 4(6-x)^2 x$

$V(x) = 4x(36-12x+x^2)$

$= 144x - 48x^2 + 4x^3$

$V'(x) = 144 - 96x + 12x^2$

$V'(x) = 12(12-8x+x^2)$

$V'(x) = 12(x-2)(x-6)$

$V'(x) = 0 \iff \boxed{x=2,}\ x=6 \notin D_V$

$X=2$

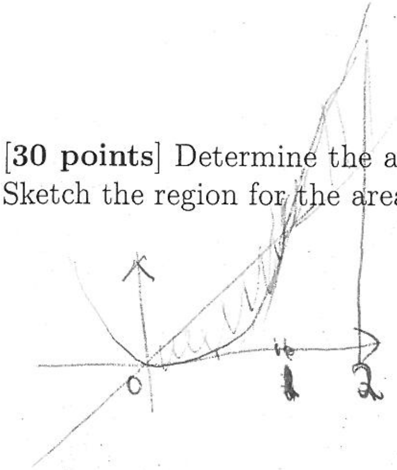
2x2 in squares

$V''(x) = -96+24x$

$V''(2) = -96+48 < 0 \Rightarrow$ max

$V(0)=0$, $V(6)=0 \Rightarrow$ there is absolute max

(22) **[30 points]** Determine the area between the curves $y = x$ and $y = x^2$ for $0 \le x \le 2$. Sketch the region for the area.



$$x = x^2$$

$$x - x^2 = 0$$

$$x(x-1) = 0$$

$$A = \int_0^1 (x - x^2)\,dx + \int_1^2 (x^2 - x)\,dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 + \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^2 = \left(\frac{1}{2} - \frac{1}{3} - 0\right) + \left(\frac{8}{3} - 2 - \frac{1}{3} + \frac{1}{2}\right)$$

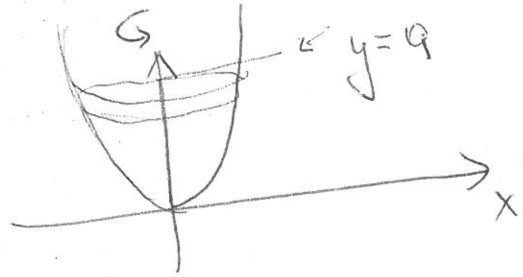
$$= \frac{1}{2} - \frac{1}{3} + \frac{8}{3} - 2 - \frac{1}{3} + \frac{1}{2} = 1 \text{ unit}^2$$

(23) **[30 points]** Find the volume of the solid obtained by rotating the region bounded by the curves $x = 2\sqrt{y}, x = 0, y = 9$ about the y-axis.

$$x = 2\sqrt{y}$$

$$x^2 = 4y$$

$$y = \frac{x^2}{4}$$



$$V = \int_0^9 \pi \left(2\sqrt{y}\right)^2 dy$$

$$= 4\pi \int_0^9 y\,dy = 4\pi \cdot \left[\frac{y^2}{2}\right]_0^9$$

$$= 4\pi \left[\frac{81}{2} - 0\right]$$

$$= 162\pi \text{ cubic units}$$