Math 728 - Homework # 1 - Peter Lewis

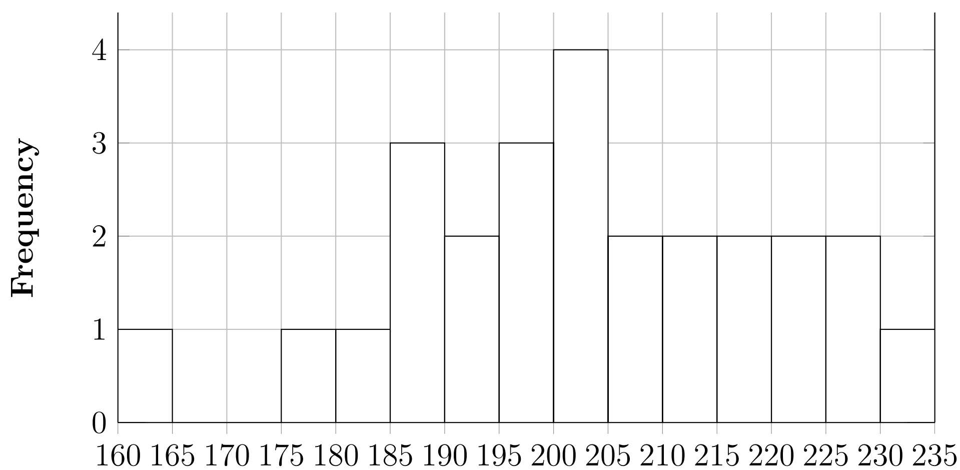
Exercise 4.1.2. The weights of 26 professional baseball pitchers are given below. Suppose we assume that the weight of a professional baseball pitcher is normally distributed with mean μ and variance σ^2.

160	175	180	185	185	185	190	190	195	195	195	200	200
200	200	205	205	210	210	218	219	220	222	225	225	232

(a) Obtain a frequency distribution and a histogram or a stem-leaf plot of the data. Use 5-pound intervals. Based on this plot, is a normal probability model credible?

Solution: Based on the above data, I have constructed the following histogram, in which the intervals of weights are of the form $[a, a+5)$.

Weights of Baseball Players



Based on this plot, a normal probability model does not seem credible.

(b) Obtain the maximum likelihood estimators of μ, σ^2, σ, and μ/σ. Locate your estimate of μ on your plot in part (a).

Solution: Let $f(x; \mu, \sigma)$ be the pdf of the normal population. Then the likelihood function is

$$
\begin{aligned}
L(\mu, \sigma; \mathbf{x}) &= \prod_{i=1}^{26} f(x_i; \mu, \sigma) \\
&= \prod_{i=1}^{26} \frac{1}{\sqrt{2\pi\sigma^2}} exp\left\{ -\frac{(x_i-\mu)^2}{2\sigma^2} \right\} \\
&= (2\pi\sigma^2)^{-13} exp\left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{26} (x_i-\mu)^2 \right\}.
\end{aligned}
$$

So, log likelihood follows as

$$l(\mu, \sigma; \mathbf{x}) = ln\big(L(\mu, \sigma; \mathbf{x})\big) = -13ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{26}(x_i - \mu)^2.$$

• Now, to find the MLE of μ, we have

$$\frac{\partial l(\mu, \sigma; \mathbf{x})}{\partial \mu} = \frac{1}{2\sigma^2}\sum_{i=1}^{26}(x_i - \mu),$$

and

$$\frac{\partial l(\mu, \sigma; \mathbf{x})}{\partial \mu} = 0 \implies \mu = \frac{1}{26}\sum_{i=1}^{26} x_i \approx 200.6.$$

Since

$$\frac{\partial^2 l(\mu, \sigma; \mathbf{x})}{\partial \mu^2} = -\frac{13}{\sigma^2} < 0,$$

$\bar{x}$ maximizes l, and hence $\hat{\mu} = \bar{X}$ is the MLE of μ.

• For the MLE of σ^2, we have

$$\frac{\partial l(\mu, \sigma; \mathbf{x})}{\partial \sigma^2} = -\frac{13}{\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{26}(x_i - \mu)^2 = 0 \implies \sigma^2 = \frac{1}{26}\sum_{i=1}^{26}(x_i - \mu)^2 \approx 293.92,$$

and

$$\frac{\partial^2 l(\mu, \sigma; \mathbf{x})}{\partial (\sigma^2)^2} = \frac{13}{\sigma^4} - \frac{1}{\sigma^6}\sum_{i=1}^{26}(x_i - \mu)^2 = \frac{1}{\sigma^4}\left[13 - \frac{1}{\sigma^2}\sum_{i=1}^{26}(x_i - \mu)^2\right].$$

If $\sigma^2 = \frac{1}{26}\sum_{i=1}^{26}(x_i - \mu)^2$, we have

$$13 - \frac{1}{\sigma^2}\sum_{i=1}^{26}(x_i - \mu)^2 = 13 - 26 < 0.$$

Hence, $\hat{\sigma^2} = \frac{1}{26}\sum_{i=1}^{26}(X_i - \mu)^2$ is the MLE of σ^2.

• By the invariance property of MLEs, we have $\hat{\sigma} = \sqrt{\frac{1}{26}\sum_{i=1}^{26}(X_i - \mu)^2} \approx 17.14$ is the MLE of σ.

• The MLE of μ/σ is approximately $200.6/17.14 \approx 11.7$

(c) Using the binomial model, obtain the maximum likelihood estimate of the proportion p of professional baseball pitchers who weigh over 215 pounds.

Solution: Notice that there are 7 players above 215 pounds. If we assume a binomial model, our likelihood function is just the pmf of Binomial(26,p):

$$L = \binom{26}{x} p^x (1-p)^{26-x}.$$

We ignore the $\binom{26}{x}$ constant, as we're maximizing in p. So, differentiate in p:

$$\begin{aligned} xp^{x-1}(1-p)^{26-x} - (26-x)(1-p)^{25-x}p^x &= p^{x-1}(1-p)^{25-x}[x(1-p) - (26-x)p] \\ &= p^{x-1}(1-p)^{25-x}[x - 26p], \end{aligned}$$

and equating to zero yields $p = x/26$.

$\implies \hat{p} = 7/26$ is the MLE of p.

(d) Determine the mle of p assuming that the weight of a professional baseball player follows the normal probability model $N(\mu, \sigma^2)$ with μ and σ unknown.

Solution: Using the MLE results in part (b), we integrate the normal pdf from 215 to infinity (by computer) to yield approximately 0.2.

Exercise 4.1.6. Show that the estimate of the pmf in expression (4.1.9) is an unbiased estimate. Find the variance of the estimator also.

Solution: The expression (4.1.9) is

$$\hat{p}(a_j) = \frac{1}{n} \sum_{i=1}^{n} I_j(X_i),$$

where

$$I_j(X_i) = \begin{cases} 1 & X_i = a_j \\ 0 & X_i \neq a_j \end{cases},$$

and the sample space is $\mathcal{D} = \{a_1, \dots, a_n\}$.

Now, we have

$$\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} I_j(X_i)\right] = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[I_j(X_i)] = \frac{1}{n}\sum_{i=1}^{n} P(X_i = a_j) = \frac{1}{n} n P(X_i = a_j) = p(a_j).$$

Hence, $\hat{p}(a_j)$ is an unbiased estimator of $p(a_j)$.

Now, the variance of the estimator is

$$Var(\hat{p}(a_j)) = Var\left(\frac{1}{n}\sum_{i=1}^{n} I_j(X_i)\right) = \frac{1}{n} Var(I_j(X_i)) = \frac{1}{n} p(a_j)\Big(1 - p(a_j)\Big).$$

Exercise 4.4.5. Let $Y_1 < Y_2 < Y_3 < Y_4$ be the order statistics of a random sample of size 4 from the distribution having pdf $f(x) = e^{-x}, 0 < x < \infty$, zero elsewhere. Find $P(Y_4 \geq 3)$.

Solution: Let X be a random variable with pdf $f(x)$. We have

$$\begin{aligned} P(Y_4 \geq 3) &= 1 - P(Y_4 < 3) \\ &= 1 - \Big(P(X < 3)\Big)^4 \\ &= 1 - \left(\int_0^3 e^{-x} dx\right)^4 \\ &= 1 - \left(1 - e^{-3}\right)^4. \end{aligned}$$

Exercise 4.4.9. Let $Y_1 < Y_2 < \ldots < Y_n$ be the order statistics of a random sample of size n from a distribution with pdf $f(x) = 1, 0 < x < 1$, zero elsewhere. Show that the kth order statistic Y_k has a beta pdf with parameters $\alpha = k$ and $\beta = n - k + 1$.

Solution: Let $g_k(y_k)$ denote the pdf of Y_k, and $F(y_k)$ the cdf of Y_k. Then, using the formula for kth order statistic pdf, we have

$$\begin{aligned} g_k(y_k) &= \begin{cases} \frac{n!}{(k-1)!(n-k)!}[F(y_k)]^{k-1}[1 - F(y_k)]^{n-k} f(y_k) & 0 < y_k < 1 \\ 0 & \text{elsewhere} \end{cases} \\ &= \begin{cases} \frac{n!}{(k-1)!(n-k)!} y_k^{k-1}(1 - y_k)^{n-k} & 0 < y_k < 1 \\ 0 & \text{elsewhere} \end{cases} \\ &= \begin{cases} \frac{\Gamma(k+n-k+1)}{\Gamma(k)\Gamma(n-k+1)} y_k^{k-1}(1 - y_k)^{n-k+1-1} & 0 < y_k < 1 \\ 0 & \text{elsewhere} \end{cases} \\ &= \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y_k^{\alpha-1}(1 - y_k)^{\beta-1} & 0 < y_k < 1 \\ 0 & \text{elsewhere} \end{cases}. \end{aligned}$$

Hence, the kth order statistic Y_k has a beta pdf with parameters $\alpha = k$ and $\beta = n - k + 1$.

Exercise 5.1.2. Let the random variable Y_n have a distribution that is $b(n, p)$.

(a) Prove that Y_n/n converges in probability to p. This result is one form of the weak law of large numbers.

Proof: Let $\{X_n\}$ be a sequence of Bernoulli random variables with parameter p. So, $\mathbb{E}[X_i] = p$, $Var(X_i) = p(1-p) < \infty$ for all $i \in \mathbb{N}$. Then, $\sum_{i=1}^n X_i = Y_n$. Hence, by the weak law of large numbers, we have that

$$\frac{Y_n}{n} = \bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i \xrightarrow{\text{P}} \mathbb{E}[X_i] = p.$$

(b) Prove that $1 - Y_n/n$ converges in probability to $1 - p$.

Proof: By part (a), we have

$$P(|1 - Y_n/n - (1-p)| > \epsilon) = P(|Y_n/n - p| > \epsilon) \xrightarrow[n\to\infty]{} 0.$$

Hence, $1 - Y_n/n \xrightarrow{\text{P}} 1 - p$.

(c) Prove that $(Y_n/n)(1 - Y_n/n)$ converges in probability to $p(1-p)$.

Proof: Using the theorem in the section, since $Y_n/n \xrightarrow{\text{P}} p$ and $(1 - Y_n/n) \xrightarrow{\text{P}} (1-p)$, we have that $Y_n/n(1 - Y_n/n) \xrightarrow{\text{P}} p(1-p)$.

Exercise 5.1.3. Let W_n denote a random variable with mean μ and variance b/n^p, where $p > 0$, μ, and b are constants (not functions of n). Prove that W_n converges in probability to μ.

Proof: Using Chebyshev's inequality, we have

$$P(|W_n - \mu| \geq \epsilon) \leq \frac{Var(W_n)}{\epsilon^2} = \frac{b/n^p}{\epsilon^2} \xrightarrow[n\to\infty]{} 0.$$

Hence, by definition of convergence in probability, $W_n \xrightarrow{\text{P}} \mu$.

Exercise 5.2.1. Let $\bar{X}_n$ denote the mean of a random sample of size n from a distribution that is $N(\mu, \sigma^2)$. Find the limiting distribution of $\bar{X}_n$.

Solution: We look at the limiting behavior of the cdf of $\bar{X}_n$. Since the sum of normal random variables is normal with sum of parameters, we have

$$\begin{aligned}
F_{\bar{X}_n}(t) &= P(X_1 + \ldots X_n \leq nt) \\
&= \int_{-\infty}^{nt} \frac{1}{\sqrt{2\pi n\sigma^2}} exp\left\{-\frac{(x - n\mu)^2}{2n\sigma^2}\right\} dx \\
&= \int_{-\infty}^{nt} \frac{1}{\sqrt{2\pi n\sigma^2}} exp\left\{-\frac{\left(\frac{x-n\mu}{\sqrt{n}\sigma}\right)^2}{2}\right\} dx.
\end{aligned}$$

Now, change variables as

$$\begin{aligned}
u &= \frac{x - n\mu}{\sqrt{n}\sigma} \\
\implies du &= \frac{1}{\sqrt{n}\sigma} dx,
\end{aligned}$$

which yields

$$\int_{-\infty}^{nt} \frac{1}{\sqrt{2\pi n\sigma^2}} exp\left\{-\frac{\left(\frac{x-n\mu}{\sqrt{n}\sigma}\right)^2}{2}\right\} dx = \int_{-\infty}^{\sqrt{n}(t-\mu)/\sigma} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$$

$$\xrightarrow[n\to\infty]{} \begin{cases} 0 & t < \mu \\ 1/2 & t = \mu \\ 1 & t > \mu \end{cases}.$$

Now, note that

$$F(t) := \begin{cases} 0 & t < \mu \\ 1 & t \geq \mu \end{cases}$$

is a cdf and $\lim_{n\to\infty} F_{\bar{X}_n}(t) = F(t)$ at every continuity point t of $F(t)$. Hence, $F_{\bar{X}_n}$ converges in distribution to a random variable that has a degenerate distribution at $t = \mu$.
