Math 728 HW 2

CHEN, Guo

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4.2.1. Let the observed value of the mean $\bar{X}$ and of the sample variance of a random sample of size 20 from a distribution that is $N(\mu, \sigma^2)$ be 81.2 and 26.5, respectively. Find respectively 90%, 95% and 99% confidence intervals for μ. Note how the lengths of the confidence intervals increase as the confidence increases.

Solution. Since $n = 20, \bar{X} = 81.2, S^2 = 26.5$, we are looking for the confidence interval

$$[\bar{X} - \frac{S}{\sqrt{n}} t_{\alpha/2,n-1},\ \bar{X} - \frac{S}{\sqrt{n}} t_{\alpha/2,n-1}].$$

If $100(1-\alpha) = 90, \alpha = 0.1, t_{0.05,19} = 1.729$, then

$$[81.2 - \sqrt{\frac{26.5}{20}} \cdot 1.729,\ 81.2 + \sqrt{\frac{26.5}{20}} \cdot 1.729] = [79.2, 83.2].$$

If $100(1-\alpha) = 95, \alpha = 0.05, t_{0.025,19} = 2.093$, then

$$[81.2 - \sqrt{\frac{26.5}{20}} \cdot 2.093,\ 81.2 + \sqrt{\frac{26.5}{20}} \cdot 2.093] = [78.8, 83.6].$$

If $100(1-\alpha) = 99, \alpha = 0.01, t_{0.005,19} = 2.861$, then

$$[81.2 - \sqrt{\frac{26.5}{20}} \cdot 2.861,\ 81.2 + \sqrt{\frac{26.5}{20}} \cdot 2.861] = [77.9, 84.5].$$

4.2.12. Let Y be $b(300, p)$. If the observed value of Y is $y = 75$, find an approximate 90% confidence interval for p.

Solution. If $100(1-\alpha) = 90, \alpha = 0.1, z_{\alpha/2} = 1.64$, and $k = 75, n = 300$, then compute

$$[\frac{k}{n} - z_{\alpha/2}\sqrt{\frac{k/n(1-k/n)}{n}},\ \frac{k}{n} + z_{\alpha/2}\sqrt{\frac{k/n(1-k/n)}{n}}],$$

we get

$$[1/4 - 1.64 \cdot \sqrt{\frac{1/4 \cdot 3/4}{300}},\ 1/4 + 1.64 \cdot \sqrt{\frac{1/4 \cdot 3/4}{300}}] = [0.209, 0.291].$$

4.2.17. It is known that a random variable X has a Poisson distribution with parameter μ. A sample of 200 observations from this distribution has a mean equal to 3.4. Construct an approximate 90% confidence interval for μ.

Solution. If X has a Poisson distribution with parameter μ, then $E(X) = \mu$ and $Var(X) = \mu$. To construct a 90% confidence interval for μ, we shall look at

$$[\bar{X} - \frac{S}{\sqrt{n}} z_{\alpha/2},\ \bar{X} + \frac{S}{\sqrt{n}} z_{\alpha/2}].$$

Since for Poisson distribution, the expectation and variance is just the parameter μ, we can use the value of the sample mean for the sample variance S^2, if we don't have the information about S^2. Hence the 90% confidence interval for μ is

$$[3.4 - \sqrt{\frac{3.4}{200}} z_{0.05},\ 3.4 + \sqrt{\frac{3.4}{200}} z_{0.05}] = [3.4 - \sqrt{\frac{3.4}{200}} \cdot 1.64,\ 3.4 + \sqrt{\frac{3.4}{200}} \cdot 1.64] = [3.186, 3.614].$$

4.2.18. Let $X_1, X_2, \cdots, X_n$ be a random sample from $N(\mu, \sigma^2)$, where both parameters μ and σ^2 are unknown. A *confidence interval* for σ^2 can be found as follows.
We know that $(n-1)S^2/\sigma^2$ is a random variable with a $\chi^2(n-1)$ distribution. Thus we can find constants a and b so that $P((n-1)S^2/\sigma^2 < b) = 0.975$ and
$P(a < (n-1)S^2/\sigma^2 < b) = 0.95$.

(a) Show that this second probability statement can be written as

$$P((n-1)S^2/b < \sigma^2 < (n-1)S^2/a) = 0.95.$$

(b) If $n = 9$ and $s^2 = 7.93$, find a 95% confidence interval for σ^2.

(c) If μ is unknown, how would you modify the preceding procedure for finding a confidence interval for σ^2?

Solution. (a)

$$a < (n-1)S^2/\sigma^2 \iff \sigma^2 < (n-1)S^2/a;$$

and

$$(n-1)S^2/\sigma^2 < b \iff \sigma^2 > (n-1)S^2/b.$$

Hence the second probability statement can be written as

$$P((n-1)S^2/b < \sigma^2 < (n-1)S^2/a) = 0.95.$$

(b) Since $\alpha = 0.05, n = 9,\ b = \chi^2_{\alpha/2,n-1} = \chi^2_{0.025,8} = 17.54$, and $a = \chi^2_{1-\alpha/2,n-1} = \chi^2_{0.975,8} = 2.18$. Also $s^2 = 7.93$. Hence the 95% confidence interval is

$$[(n-1)S^2/b,\ (n-1)S^2/a] = [\frac{8\cdot 7.93}{17.54}, \frac{8\cdot 7.93}{2.18}] = [3.62, 29.10].$$

(c) In this case, define $\tilde{S}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2$, then $\frac{n\tilde{S}^2}{\sigma^2} \sim \chi^2(n)$. We can follow similar procedure by finding a and b from $P(n\tilde{S}^2/\sigma^2 < b) = 0.975$ and $P(a < n\tilde{S}^2/\sigma^2 < b) = 0.95$. Then the new confidence interval should be $[n\tilde{S}^2/b,\ n\tilde{S}^2/a]$.

4.2.21. Let two independent random samples, each of size 10, from two normal distributions $N(\mu_1,\sigma^2)$ and $N(\mu_2,\sigma^2)$ yields $\bar{x} = 4.8, s_1^2 = 8.64, \bar{y} = 5.6, s_2^2 = 7.88$. Find a 95% confidence interval for $\mu_1 - \mu_2$.

Solution. We'll use the formula to find the confidence interval

$$((\bar{x}-\bar{y}) - t_{\alpha/2,n-2}s_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}, (\bar{x}-\bar{y}) + t_{\alpha/2,n-2}s_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}})$$

With information we get

$$s_p^2 = \frac{9\cdot 8.64 + 9\cdot 7.88}{9+9} = 8.26,$$

So the 95% confidence interval for $\mu_1 - \mu_2$ is

$$(4.8 - 5.6 - t_{0.025,18}\cdot 2.874\cdot 0.447, 4.8 - 5.6 + t_{0.025,18}\cdot 2.874\cdot 0.447) = (-3.499, 1.899)$$

4.2.22. Let two independent random variables, Y_1s and Y_2, with binomial distributions that have parameters $n_1 = n_2 = 100$, p_1, and p_2, respectively, be observed to be equal to $y_1 = 50$ and $y_2 = 40$. Determine an approximate 90% confidence interval for $p_1 - p_2$.

Solution. Let $\hat{p}_1 = \frac{50}{100} = 0.5, \hat{p}_2 = \frac{40}{100} = 0.4$, then an 95% confidence interval for $p_1 - p_2$ is

$$(\hat{p}_1 - \hat{p}_2 - z_{0.05}\cdot\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1}+\frac{\hat{p}_2(1-\hat{p}_2)}{n_2}},\ \hat{p}_1 - \hat{p}_2 + z_{0.05}\cdot\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1}+\frac{\hat{p}_2(1-\hat{p}_2)}{n_2}})$$

So the confidence interval is

$$(0.5-0.4-1.64\sqrt{\frac{0.5\cdot 0.5}{100}+\frac{0.4\cdot 0.6}{100}}, 0.5-0.4+1.64\sqrt{\frac{0.5\cdot 0.5}{100}+\frac{0.4\cdot 0.6}{100}}) = (-0.0148, 0.2148).$$

4.2.27. Let $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_m$ be two independent random samples from the respective normal distribution $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$, where the four parameters are unknown. To construct a *confidence interval for the ratio,* σ_1^2/σ_2^2, of the variances, form the quotient of the two independent χ^2 variables, each divided by its degrees of freedom, namely,

$$F = \frac{\frac{(m-1)S_2^2}{\sigma_2^2}/(m-1)}{\frac{(n-1)S_2^2}{\sigma_2^2}/(n-1)} = \frac{S_2^2/\sigma_2^2}{S_1^2/\sigma_1^2},$$

where S_1^2 and S_2^2 are the respective sample variances.
(a) What kind of distribution does F have?
(b) From the appropriate table, a and b can be found so that $P(F < b) = 0.975$ and $P(a < F < b) = 0.95$.
(c) Rewrite the second probability statement as

$$P\Big[a\frac{S_1^2}{S_2^2} < \frac{\sigma_1^2}{\sigma_2^2} < b\frac{S_1^2}{S_2^2}\Big] = 0.95.$$

The observed values, s_1^2 and s_2^2, can be inserted in these inequalities to provide a 95% confidence interval for σ_1^2/σ_2^2.

Solution. (a) This is F-distribution with parameter $m-1$ and $n-1$.

(b)& (c) Note

$$\frac{S_2^2/\sigma_2^2}{S_1^2/\sigma_1^2} < b \iff \frac{\sigma_1^2}{\sigma_2^2}\cdot\frac{S_2^2}{S_1^2} < b \iff \frac{\sigma_1^2}{\sigma_2^2} < b\frac{S_1^2}{S_2^2},$$

and

$$\frac{S_2^2/\sigma_2^2}{S_1^2/\sigma_1^2} > a \iff \frac{\sigma_1^2}{\sigma_2^2}\cdot\frac{S_2^2}{S_1^2} > a \iff \frac{\sigma_1^2}{\sigma_2^2} > a\frac{S_1^2}{S_2^2}.$$

That's why we can write $P(a < \frac{S_2^2/\sigma_2^2}{S_1^2/\sigma_1^2} < b) = 0.95$ as

$$P\Big(a\frac{S_1^2}{S_2^2} < \frac{\sigma_1^2}{\sigma_2^2} < b\frac{S_1^2}{S_2^2}\Big) = 0.95.$$