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Statistics **Math 728** **Homework 2**

4.2.1 Let the observed value of the mean $\overline{X}$ and of the sample variance of a random sample of size 20 from a distribution that is $N(\mu, \sigma^2)$ be 81.2 and 26.5, respectively. Find respectively 90%, 95%, and 99% confindence intervals for μ.

Solution. We use algebra derivation given in Example 4.2.1.
For 90%: $1-\alpha = .9 \implies \alpha = .1$. There are $20-1 = 19$ degrees of freedom. Thus the lower limit is $81.2-(1.729)(\frac{\sqrt{26.5}}{\sqrt{20}}) = 79.209$ and the uppr limit is $81.2+(1.729)(\frac{\sqrt{26.5}}{\sqrt{20}}) = 83.190$. So $(79.21, 83.19)$ is the interval with 90% confidence.
For 95%: $1-\alpha = .95 \implies \alpha = .05$. There are $20-1 = 19$ degrees of freedom. Thus the lower limit is $81.2-(2.093)(\frac{\sqrt{26.5}}{\sqrt{20}}) = 78.791$ and the uppr limit is $81.2+(2.093)(\frac{\sqrt{26.5}}{\sqrt{20}}) = 83.609$. So $(78.79, 83.61)$ is the interval with 95% confidence.
For 99%: $1-\alpha = .99 \implies \alpha = .01$. There are $20-1 = 19$ degrees of freedom. Thus the lower limit is $81.2-(2.861)(\frac{\sqrt{26.5}}{\sqrt{20}}) = 77.907$ and the uppr limit is $81.2+(2.861)(\frac{\sqrt{26.5}}{\sqrt{20}}) = 84.493$. So $(77.91, 84.49)$ is the interval with 95% confidence. $\triangle$

4.2.12 Let Y be $b(300, p)$. If the observed value of Y is $y = 75$, find an approximate 90% confidence interval for p.

Solution. We have $n = 300$ and a porportion $p = 75/300 = .25$. We have $1-\alpha = .90 \implies \alpha = .1$ Thus $z_{\alpha/2} = 1.645$. Thus the lower limit is $.25-(1.645)(\frac{\sqrt{.25(1-.25)}}{\sqrt{300}}) = .209$ and the upper limit is $.25+(1.645)(\frac{\sqrt{.25(1-.25)}}{\sqrt{300}}) = .291$. Hence we have a 90% confidence interval for $(.21, .29)$. $\triangle$

4.2.17 It is known that a random variable X has Poisson distribution with parameter μ. A sample of 200 observation from this distribution has a mean equal to 3.4. Construct an approximate 90% confidence interval for μ.

Solution. Since X has the Poisson distribution the mean of X is equal to the variance. We have $1-\alpha = .9 \implies \alpha = .1$. Thus $z_{\alpha/2} = 1.645$. Then the lower limit is $3.4-(1.645)(\frac{\sqrt{3.4}}{\sqrt{200}}) = 3.186$ and the upper limit is $3.4+(1.645)(\frac{\sqrt{3.4}}{\sqrt{200}}) = 3.614$. Hence we have a 90% confidence interval for $(3.19, 3.61)$. $\triangle$

4.2.18 Let $X_1, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$, where both parameters μ and σ are unkown. A confidence interval for σ^2 can be found as follows. We know that $(n-1)S^2/\sigma^2$ is a random variable with $\chi^2(n-1)$ distribution. Thus we can find constants a and b so that $P((n-1)S^2/\sigma^2 < b) = 0.975$ and $P(a < (n-1)S^2/\sigma^2 < b) = 0.95$.

(a) Show that this second probability statement can be written as
$$P((n-1)S^2/b < \sigma^2 < (n-1)S^2/a) = 0.95.$$

Proof. We have
$$a < \frac{(n-1)S^2}{\sigma^2} < b \iff \frac{1}{b} < \frac{\sigma^2}{(n-1)S^2} < \frac{1}{a} \iff \frac{(n-1)S^2}{b} < \sigma^2 < \frac{(n-1)S^2}{a}.$$
Thus $P(a < (n-1)S^2/\sigma^2 < b) = 0.95 \iff P((n-1)S^2/b < \sigma^2 < (n-1)S^2/a) = 0.95$. $\square$

(b) If $n = 9$ and $s^2 = 7.93$, find a 95% confidence interval for σ^2.

Solution. If $n = 9$ and $s^2 = 7.93$, then $(n-1)S^2$ is $(9-1)(7.93) = 63.44$, $a = 2.180$, and $b = 17.535$. Thus the lower limit is $63.44/17.535 = 3.618$ and the upper limit is $63.44/2.18 = 29.101$. Thus we have 95% confidence for the interval $(3.62, 29.10)$. $\triangle$

(c) If μ is known, how would you modify the proceeding procedure for finding a confidence interval σ^2?

Solution. We know that $\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2$ is $\chi^2(n)$. Then we use $\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2$ in place of $(n-1)S^2/\sigma^2$. △

4.2.21 Let two independent random samples, each of size 10, from two normal distributions $N(\mu_1,\sigma^2)$ and $N(\mu_2,\sigma^2)$ yield $\overline{x}=4.8$, $s_1^2=8.64$, $\overline{y}=5.6$, $s_2^2=7.88$. Find a 95% confidence interval for $\mu_1-\mu_2$.

Solution. We use equation 4.2.13. We have $n_1=n_1=10$. Thus $n=n_1+n_2=20$. We have $1-\alpha=.95 \implies \alpha=.05$. Then $t_{.025,18}=2.101$. Thus the lower limit is

$$(4.8-5.6)-(2.101)\sqrt{\frac{(9)(8.64)+(9)(7.88)}{18}}\sqrt{\frac{1}{10}+\frac{1}{10}}=-3.5$$

and the upper limit is

$$(4.8-5.6)+(2.101)\sqrt{\frac{(9)(8.64)+(9)(7.88)}{18}}\sqrt{\frac{1}{10}+\frac{1}{10}}=1.9.$$

Thus we have a 95% confidence interval for $(-3.5,1.9)$. △

4.2.22 Let two independent random variables, Y_1 and Y_2, with binomial distributions that have parameters $n_1=n_2=100$, p_1, and p_2, respectively, be observed to be equal to $y_1=50$ and $y_2=40$. Determine an approximate 90% confidence interval for p_1-p_2.

Solution. We use equation 4.2.14. We have $p_1=50/100=.5$ and $p_2=40/100=.4$. We find $z_{\alpha/2}$ for $\alpha=.1$. Then $z_{\alpha/2}=1.645$. Thus the lower limit is

$$(.5-.4)-(1.645)\sqrt{\frac{(.5)(.5)}{100}+\frac{(.4)(.6)}{100}}=-.015$$

and the upper limit is

$$(.5-.4)-(1.645)\sqrt{\frac{(.5)(.5)}{100}+\frac{(.4)(.6)}{100}}=.215.$$

Thus we have a 90% confidence interval for $(-.02,.22)$. △

4.2.27 Let $X_1,\ldots,X_n$ and $Y_1,\ldots,Y_m$ be two independent random samples from the respective normal distributions $N(\mu_1,\sigma_1^2)$ and $N(\mu_2,\sigma_2^2)$, where the four parameters are unknown. To construct a confidence interval for the ratio, σ_1^2/σ_2^2, of the variances, form the quotient of the two independent χ^2 variables, each divided by its degrees of freedom, namely,

$$F=\frac{\frac{(m-1)S_2^2}{\sigma_2^2}/(m-1)}{\frac{(n-1)S_1^2}{\sigma_1^2}/(n-1)},$$

where S_1^2 and S_2^2 are the respective sample variances.

(a) What kind of distribution does F have?

Solution. $U=(m-1)S_2^2$ has $\chi^2(m-1)$ distribution and $V=(n-1)S_1^2$ has $\chi^2(n-1)$ distribution. Thus be equation 3.6.7 in the text

$$F=\frac{U/r_1}{V/r_2},$$

where $r_1=m-1$ and $r_2=n-1$, has F-distribution. △

(b) From the appropriate table, a and b can be found so that $P(F<b)=0.975$ and $P(a<F<b)=0.95$.

Solution. For $P(F<b)=0.975$, $b=F_{.025}(m-1,n-1)$. For $P(a<F<b)=0.95$, $b=F_{.05}(m-1,n-1)$ and $a=1/b$. △

(c) Rewrite the second probability statement as

$$P\left[a\frac{S_1^2}{S_2^2} < \frac{\sigma_1^2}{\sigma_2^2} < b\frac{S_1^2}{S_2^2}\right] = 0.95.$$

The observed values, s_1^2 and s_2^2, can be inserted in these inequalities to provide a 95% confidence interval for σ_1^2/σ_2^2.

Solution. We have

$$F = \frac{\frac{(m-1)S_2^2}{\sigma_2^2}/(m-1)}{\frac{(n-1)S_1^2}{\sigma_1^2}/(n-1)} = \frac{\frac{(m-1)S_2^2}{(m-1)\sigma_2^2}}{\frac{(n-1)S_1^2}{(n-1)\sigma_1^2}} = \frac{S_2^2\sigma_1^2}{S_1^2\sigma_2^2}.$$

Thus

$$a < \frac{S_2^2\sigma_1^2}{S_1^2\sigma_2^2} < b \iff a\frac{S_1^2}{S_2^2} < \frac{\sigma_1^2}{\sigma_2^2} < b\frac{S_1^2}{S_2^2}.$$

Hence

$$P\left[a\frac{S_1^2}{S_2^2} < \frac{\sigma_1^2}{\sigma_2^2} < b\frac{S_1^2}{S_2^2}\right] = 0.95.$$

$\triangle$