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Math 728
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Homework 3

Problem 4.5.8: *Assume the life of a tire given by X is distributed $N(\theta, 5000^2)$ Past experience indicates that $\theta = 30000$. The manufacturere claims the tires made by a new process have mean $\theta > 30000$. Is it possible that $\theta = 35000$? Tets this claim by testing $H_0 : \theta = 30000$ against $H_1 : \theta > 30000$. We observe n independent values denoted $x_1, x_2, \dots, x_n$ and we reject H_0 iff $\bar{x} \geq c$. Determine n and c such that the power function of the test, $\gamma(\theta)$ has the values $\gamma(30000) = 0.01$ and $\gamma(35000) = 0.98$*

Solution: First note that the power function of the test is given as follows,

$$\begin{aligned}\gamma(\theta) &= P(\bar{x} \geq c \mid \mu = \theta, \sigma = 5000) \\ &= P\left(\frac{\bar{x} - \theta}{5000/\sqrt{n}} \geq \frac{c - \theta}{5000/\sqrt{n}}\right)\end{aligned}$$

Now because $\frac{\bar{x}-\theta}{5000/n} \sim N(0,1)$, it follows that in order to satisfy the given conditions,

$$\frac{c - 30000}{5000/\sqrt{n}} = z_{0.01} \quad \text{and} \quad \frac{c - 35000}{5000/\sqrt{n}} = z_{0.98}$$

All that remains is to substitute in the appropriate z values and solve the system of equations as follows,

$$\begin{aligned}\frac{c - 30000}{5000/\sqrt{n}} &= 2.326 \\ c - 30000 &= \frac{5000(2.326)}{\sqrt{n}} \\ c &= \frac{11630}{\sqrt{n}} + 30000 \qquad (1)\end{aligned}$$

$$\begin{aligned}\frac{c - 35000}{5000/\sqrt{n}} &= -2.054 \\ c - 35000 &= \frac{5000(-2.054)}{\sqrt{n}} \\ c &= \frac{-10270}{\sqrt{n}} + 35000 \qquad (2)\end{aligned}$$

Setting (1) equal to (2) yeilds,

$$\begin{aligned}\frac{-10270}{\sqrt{n}} + 35000 &= \frac{11630}{\sqrt{n}} + 30000 \\ 5000 &= \frac{21900}{\sqrt{n}} \\ \sqrt{n} &= \frac{21900}{5000} \\ n &= (4.38)^2 = 19.1844 \qquad (3)\end{aligned}$$

In order to be conservative rounding (3) to 20 and substituting into (1) yields,

$$c = \frac{11630}{\sqrt{20}} + 30000 = 32600.547$$

So as a conservative decision, $n = 20$ and $c = 32600.547$

Problem 4.5.9: *let X have a Poisson distribution with mean θ. Consider the simple hypothesis $H_0 : \theta = 1/2$ and the alternative composite hypothesis $H_1 : \theta < 1/2$ Thus, $\Omega = \{\theta : 0 < \theta \leq 1/2\}$. Let $X_1, X_2, \ldots, X_{12}$ be a random sample from this distribution. We reject H_0 iff the observed value of $Y = X_1 + X_2 + \cdots + X_{12} \leq 2$. If $\gamma(\theta)$ is the power function of the test, find the powers $\gamma(1/2), \gamma(1/3), \gamma(1/4), \gamma(1/6),$ and $\gamma(1/12)$. Sketch the graph of $\gamma(\theta)$. What is the significance level of the test?*

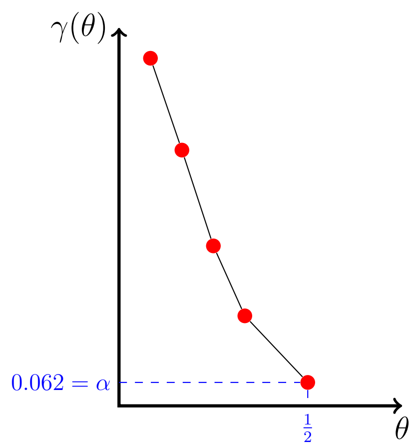
Solution: First note that because Y is a sum of i.i.d poisson random variables $Y \sim \text{Poisson}(12\theta)$ and $\gamma(\theta)$ is given as follows,

$$\begin{aligned}\gamma(\theta) &= P(Y \leq 2 \mid \mu = \theta) \\ &= \sum_{i=0}^{2} \frac{e^{-12\theta}(12\theta)^i}{i!} \\ &= e^{-12\theta} + e^{-12\theta}(12\theta) + \frac{e^{-12\theta}(12\theta)^2}{2}\end{aligned}$$

The computed values of $\gamma(\theta)$ for desired values of θ are summarized below:

θ	$1/2$	$1/3$	$1/4$	$1/6$	$1/12$
$\gamma(\theta)$	$\frac{25}{e^6}$	$\frac{13}{e^4}$	$\frac{17}{2e^3}$	$\frac{5}{e^2}$	$\frac{5}{2e}$

The graph with these values shown is below:



Finally the significance of the test is given by the probability of rejecting H_0 when it is true as expressed by,

$$P(Y \leq 2 \mid \theta = 1/2) = 0.062$$

Problem 4.5.10: *Let Y have a binomial distribution with parameters n and p. We reject $H_0 : p = 1/2$ and accept $H_1 : p > 1/2$ if $Y \geq c$. Find n and c to give a power function $\gamma(p)$ such that $\gamma(1/2) = 0.10$ and $\gamma(2/3) = .95$, approximately.*

Solution: First note the power function is given by,

$$\begin{aligned}
\gamma(p) &= P(Y \geq c \mid p = \hat{p}) \\
&= 1 - P(Y < c \mid p = \hat{p}) \\
&= 1 - \sum_{k=0}^{\lceil c \rceil - 1} \binom{n}{k} p^k (1-p)^{n-k}
\end{aligned}$$

However, because this is computationally difficult to solve, we instead use the normal approximation to the binomial as follows,

$$\begin{aligned}
\gamma(p) &= P(Y \geq c \mid p = \hat{p}) \\
&\approx P(X \geq c) \text{ where } X \sim N(np, np(1-p)) \\
&= P\left(\frac{X - np}{\sqrt{np(1-p)}} \geq \frac{c - np}{\sqrt{np(1-p)}} \right)
\end{aligned}$$

Because $\frac{X-np}{\sqrt{np(1-p)}} \sim N(0,1)$ it follows that in order to satisfy the conditions,

$$\frac{c - n(0.5)}{\sqrt{n(0.25)}} = z_{0.10} \quad \text{and} \quad \frac{c - n(2/3)}{\sqrt{n(2/9)}} = z_{0.95}$$

All that remains is to substitute in the appropriate z values and solve the system of equations as follows,

$$\begin{aligned}
\frac{c - n(0.5)}{\sqrt{n(0.25)}} &= z_{0.10} \\
\frac{c - n(0.5)}{\sqrt{n(0.25)}} &= 1.282 \\
c - 0.5n &= 1.282\sqrt{n(0.25)} \\
c &= 1.282\sqrt{n(0.25)} + 0.5n \qquad (4)
\end{aligned}$$

$$\begin{aligned}
\frac{c - n(2/3)}{\sqrt{n(2/9)}} &= z_{0.95} \\
\frac{c - n(2/3)}{\sqrt{n(2/9)}} &= -1.645 \\
c - n(2/3) &= -1.645\sqrt{n(2/9)} \\
c &= -1.645\sqrt{n(2/9)} + n(2/3) \qquad (5)
\end{aligned}$$

Setting (4) equal to (5) yields,

$$\begin{aligned} -1.645\sqrt{n(2/9)} + n(2/3) &= 1.282\sqrt{n(0.25)} + 0.5n \\ n\left[(2/3) - (1/2)\right] &= \sqrt{n}[1.282\sqrt{0.25} + 1.645\sqrt{2/9}] \\ \sqrt{n} &= \frac{1.282\sqrt{0.25} + 1.645\sqrt{2/9}}{(2/3) - (1/2)} \\ n &= (8.50)^2 = 72.25 \qquad (6) \end{aligned}$$

In order to be conservative rounding (6) to 73 and substituting into (4) yields,

$$c = 1.282\sqrt{73(0.25)} + 0.5(73) = 41.98$$

Rounding to 42 in order to be conservative gives $n \approx 73$ and $c \approx 42$

Problem 4.5.13: *Let p denote the probability that, for a particular tennis player, the first serve is good. Since p=0.40, this player decided to take lessons in order to increase p. When the lessons are completed, the hypothesis $H_0 : p = 0.40$ is tested against $H_1 : p > 0.40$ based on $n = 25$ trials. Let y equal the number of first serves that are good, and let the critical region be defined by $C = \{y : y \geq 13\}$. Solve (a) and (b).*

Solution:

(a) *Determine $\alpha = P(Y \geq 13 \mid p = 0.40)$.*

This follows directly from the cumulative distribution function of Y,

$$\begin{aligned} P(Y \geq 13 \mid p = 0.40) &= 1 - P(Y < 13 \mid p = 0.40) \\ &= 1 - \sum_{k=0}^{12} \binom{25}{k} (0.4)^k (0.6)^{25-k} \\ &= 1 - 0.846 \\ &= 0.154 \end{aligned}$$

(b) *Find $\beta = P(Y < 13 \mid p = 0.60)$ so that $1 - \beta$ is the power at $p = 0.60$.*

This similarly follows directly from the cumulative distribution function of Y,

$$\begin{aligned} P(Y < 13 \mid p = 0.60) &= \sum_{k=0}^{12} \binom{25}{k} (0.6)^k (0.4)^{25-k} = 0.154 \\ 1 - \beta &= 0.846 \end{aligned}$$

Problem 4.6.4: *Consider the one-sided t-test for $H_0 : \mu = \mu_0$ versus $H_{A1} : \mu > \mu_0$ constructed in Example 4.5.4 and the two-sided t-test for $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$ given in (4.6.9). Assume that both tests are of size α. Show that for $\mu > \mu_0$, the power function of the one-sided test is larger than the power function of the two-sided test.*

Solution: This follows from direct comparison of the power functions as follows. First denote the power function for the first test by $\gamma_1(\mu)$ and denote the power function of the second test by $\gamma_2(\mu)$. The power functions then follow from their definitions.

$$\begin{aligned}
\gamma_1(\mu) &= P\left(\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} \geq t_\alpha(n-1) \mid \mu\right) \\
&= P\left(\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} \geq t_{\alpha/2}(n-1) \mid \mu\right) + P\left(t_{\alpha/2} > \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} \geq t_\alpha(n-1) \mid \mu\right) \\
\gamma_2(\mu) &= P\left(\left|\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\right| \geq t_{\alpha/2}(n-1) \mid \mu\right) \\
&= P\left(\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} \geq t_{\alpha/2}(n-1) \mid \mu\right) + P\left(\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} \leq -t_{\alpha/2}(n-1) \mid \mu\right)
\end{aligned}$$

Therefore, for $\mu > \mu_0$, $\gamma_1(\mu) > \gamma_2(\mu)$ iff,

$$P\left(\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} \leq -t_{\alpha/2}(n-1) \mid \mu\right) < P\left(t_{\alpha/2} > \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} \geq t_\alpha(n-1) \mid \mu\right)$$

If the null is true and $\mu_0 = \mu$ then it follows that,

$$P\left(\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} \leq -t_{\alpha/2}(n-1) \mid \mu\right) = \frac{\alpha}{2} = P\left(t_{\alpha/2} > \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} \geq t_\alpha(n-1) \mid \mu\right)$$

However, if $\mu > \mu_0$, because the true distribution is symmetric about μ but the t-values are symetric around μ_0, it follows that the interval on the RHS is closer to the true mean while the interval on the LHS is further away from the true mean. Based on the symmetry of the distribution about μ the desired inequality follows.

Problem 4.6.5: *Assume that the weight of cereal in a "10-ounce box" is* $N(\mu, \sigma^2)$*. To test* $H_0 : \mu = 10.1$ *against* $H_1 : \mu > 10.1$*, we take a random sample of size* $n = 16$ *and observe that* $\bar{x} = 10.4$ *and* $s = 0.4$*. Solve (a) and (b).*

Solution:

(a) *Do we accept or reject* H_0 *at the 5% significance level?*

Because σ is unknown and the sample size is small, we utilize a $t - test$. Therefore, we reject H_0 iff

$$T = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \geq t_{0.05}(n-1)$$

From the observed values,

$$T = \frac{10.4 - 10.1}{0.4/4} = 3 \text{ and } t_{0.05}(15) = 1.753$$

Therefore we reject H_0 in favor of H_1

(b) *What is the approximate p-value of this test?*

By definition the p-value of the test is given by,

$$p = P(T \geq 3)$$

Because $T \sim T(15)$ it follows from the table that,

$$p = P(T \geq 3) \approx 0.005$$

Problem 4.6.7: *Among the data collected for the WHO air quality monitoring project is a measure of suspended particles in* $\mu g/m^3$*. Let X and Y equal the concentration of suspended particles in* $\mu g/m^3$ *in the city center for Melbourne and Houston respectively. Using* $n = 13$ *observations of X and* $m = 16$ *observations of Y, we test* $H_0 : \mu_X = \mu_Y$ *against* $H_1 : \mu_X < \mu_Y$*. Solve (a) and (b).*

Solution:

(a) *Define the test statistic and critical region, assuming the unknown variances are equal. Let* $\alpha = 0.05$

Because the sample is small and the variances are unknown but assumed to be the same, we use a t-test with a pooled variance estimator, which is denoted S_p^2, where we reject iff

$$T = \frac{(\bar{y} - \bar{x}) - (\mu_{0,Y} - \mu_{0,X})}{S_p\sqrt{\frac{1}{m} + \frac{1}{n}}} \geq t_\alpha(n+m-2) \text{ where } S_p = \sqrt{\frac{(n-1)S_X^2 + (m-1)S_Y^2}{n+m-2}}$$

Which simplifies to:

$$T = \frac{\bar{y} - \bar{x}}{S_p\sqrt{\frac{1}{16} + \frac{1}{13}}} \geq 1.703 \text{ where } S_p = \sqrt{\frac{(12)S_X^2 + (15)S_Y^2}{27}}$$

(b) *If* $\bar{x} = 72.9, s_x = 25.6, \bar{y} = 81.7,$ *and* $s_y = 28.3,$ *calculate the value of the test statistic and state your conclusions.* This follows from simply substituting in the observed values. First S_p is given as follows,

$$S_p = \sqrt{\frac{(12)(25.6)^2 + (15)(28.3)^2}{27}} = 27.133$$

Then substituting this value yields,

$$T = \frac{81.7 - 72.9}{27.133\sqrt{\frac{1}{16} + \frac{1}{13}}} = 0.868 \not\geq 1.703$$

Therefore we do not reject the null, H_0

Problem 4.6.8: *Let p equal the proportion of drivers who use a seatbelt in a country that does not have a mandatory seat belt law. It was claimed that p=0.14. An advertising campaign was conducted to increase this proportion. Two months after the campaign, y=104 out of the random sample of n=590 drivers were wearing their seat belts. Was the campaign successful? Answer (a)-(c)*

Solution:

(a) *Define the null and alternative hypotheses.*

First denote $p_0 = 0.14$ and $p = Y/n$. Because we want to be conservative, we assume nothing has changed letting $H_0 : p_0 = p_1$ be the null hypotheses. Then because we want to test only if there was improvement we let $H_1 : p_0 < p_1$ be the alternative hypothesis.

(b) *Define a critical region with an* $\alpha = 0.01$ *significance level.*

Because p is the sample mean of a series of Bernoulli random variables we can use a Z test where we reject H_0 if and only if,

$$Z = \frac{p - p_0}{\sqrt{\frac{p(1-p)}{n}}} \geq z_{0.01} \text{ or with desired values: } Z = \frac{p - p_0}{\sqrt{\frac{p(1-p)}{590}}} \geq 2.326$$

(c) *Determine the approximate p-value and state your conclusion*

First substituting observed values from the data yields,

$$\begin{aligned} Z &= \frac{104/590 - 0.14}{\sqrt{\frac{(104/590)(1-104/590)}{590}}} \\ &= \frac{0.0363}{0.0157} \\ &= 2.3121 \end{aligned}$$

Because $2.3121 < 2.326$ we fail to reject $H_0 : p_0 = p_1$ at the significance level $\alpha = 0.01$. Given this observed Z test statistic the p-value can be found. Because $Z \sim N(0,1)$, by definition the p-value of the test is given by:

$$\begin{aligned} p - value &= P(Z \geq 2.3121) \\ &\approx 1 - 0.9896 = 0.0104 \end{aligned}$$

Problem 4.6.9: *In Exercise 4.2.18 we found a confidence interval for the variance σ^2 using the variance S^2 of a random sample n arising from $N(\mu, \sigma^2)$, where the mean is unknown. In testing $H_0 : \sigma^2 = \sigma_0^2$ against $H_1 : \sigma^2 > \sigma_0^2$, use the critical region defined by $(n-1)S^2/\sigma_0^2 \geq c$. That is, reject H_0 in favor of H_1 if $S^2 \geq c\sigma_0^2/(n-1)$. If $n = 13$ and the significance level $\alpha = 0.025$, determine c.*

Solution: First note that given the desired values, we are seeking to find is c such that:

$$P\left(\frac{(12)S^2}{\sigma_0^2} \geq c\right) = 0.025$$

Because it is known that $12S^2/\sigma_0^2 \sim \chi^2(12)$ all that remains is to utilize a table to find the appropriate value. It follows that $c = 23.337$

Problem 4.6.10: *In Exercise 4.2.27, in finding a confidence interval for the ratio of the variances of two normal distributions, we used a statistic S_1^2/S_2^2, which has an F-distribution when those two variances are equal. If we denote the statistic by F, we can test $H_0 : \sigma_1^2 = \sigma_2^2$ against $H_1 : \sigma_1^2 > \sigma_2^2$ using the critical region $F \geq c$. If $n = 13, m = 11,$ and $\alpha = 0.05$, find c.*

Solution: First note that given the desired values we are seeking to find c such that,

$$P\left(\frac{S_1^2}{S_2^2} \geq c\right) = 0.05$$

Because it is known that $S_1^2/S_2^2 \sim F(10, 12)$ all that remains is to utilize a table to find the appropriate value. It follows that $c = 2.75$.