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All questions taken from the book.

5.1.3 Let W_n denote a random variable with mean μ and variance b/n^p, where $p > 0$, μ, and b are constants (not functions of n). Prove that W_n converges in probability to μ.

Proof. Let $\varepsilon > 0$. We have

$$\lim_{n\to\infty} P[|W_n - \mu| \geq \varepsilon] \leq \lim_{n\to\infty} \frac{E[(W_n - \mu)^2]}{\varepsilon^2} = \lim_{n\to\infty} \frac{\text{Var}(W_n)}{\varepsilon^2} = \lim_{n\to\infty} \frac{b}{\varepsilon^2 n^p} = 0,$$

using Chebychev's inequality where $g(x) = x^2$. Hence $W_n \xrightarrow{\text{P}} \mu$. □

4.2.12 Let Y be $b(300, p)$. If the observed value of Y is $y = 75$, find an approximate 90% confidence interval for p.

Solution. We have $n = 300$ and a porportion $p = 75/300 = .25$. We have $1 - \alpha = .90 \implies \alpha = .1$ Thus $z_{\alpha/2} = 1.645$. Thus the lower limit is $.25 - (1.645)(\frac{\sqrt{.25(1-.25)}}{\sqrt{300}}) = .209$ and the upper limit is $.25 + (1.645)(\frac{\sqrt{.25(1-.25)}}{\sqrt{300}}) = .291$. Hence we have a 90% confidence interval for $(.21, .29)$. △

4.2.21 Let two independent random samples, each of size 10, from two normal distributions $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$ yield $\overline{x} = 4.8$, $s_1^2 = 8.64$, $\overline{y} = 5.6$, $s_2^2 = 7.88$. Find a 95% confidence interval for $\mu_1 - \mu_2$.

Solution. We use equation 4.2.13. We have $n_1 = n_1 = 10$. Thus $n = n_1 + n_2 = 20$. We have $1 - \alpha = .95 \implies \alpha = .05$. Then $t_{.025,18} = 2.101$. Thus the lower limit is

$$(4.8 - 5.6) - (2.101)\sqrt{\frac{(9)(8.64) + (9)(7.88)}{18}}\sqrt{\frac{1}{10} + \frac{1}{10}} = -3.5$$

and the upper limit is

$$(4.8 - 5.6) + (2.101)\sqrt{\frac{(9)(8.64) + (9)(7.88)}{18}}\sqrt{\frac{1}{10} + \frac{1}{10}} = 1.9.$$

Thus we have a 95% confidence interval for $(-3.5, 1.9)$. △

4.6.08 Let p equal the proportion of drivers who use a seat belt in a country that does not a mandatory seat belt law. It was claimed that $p = .14$. An advertising campaign was conducted to increase this proportion. Two months after the campaign, $y = 104$ out of a random sample of $n = 590$ drivers were wearing their seat belts. Was the campaign succesful?

(a) Define the null and alternative hypotheses.

Solution. Define $H_0 : p = .14$ and $H_1 : p > .14$. △

(b) Define a critical region with an $\alpha = .01$ significance level.

Solution. The critical region is

$$C = \{z : z \geq 2.326\},$$

where

$$z = \frac{\frac{y}{n} - .14}{\sqrt{(.14)(.86)}}\sqrt{n}.$$

△

(c) Determine the approximate p-value and state your conclusion.

Solution. We have the observed value of

$$z = \frac{\frac{104}{590} - .14}{\sqrt{(.14)(.86)}}\sqrt{590} = 2.539.$$

Thus H_0 is rejected since $2.539 > 2.326$. We conclude that the campaign was succesful. △

6.1.02 Let $X_1, X_2, \ldots, X_n$ be a random sample from each of the distributions have the following pdfs:
(a) $f(x;\theta) = \theta x^{\theta-1}$, $0 < x < 1$, $0 < \theta < \infty$, zero elsewhere.
(b) $f(x;\theta) = e^{-(x-\theta)}$, $0 \le x < \infty$, $-\infty < \theta < \infty$, zero elsewhere. Note this ia a nonregular case.
In each case find the mle $\hat{\theta}$ of θ.

Solution. (a) We have

$$L(\theta) = \prod_{i=1}^{n} \theta x_i^{\theta-1}.$$

Then

$$\begin{aligned} l(\theta) = \log[L(\theta)] = \log\left(\prod_{i=1}^{n} \theta x_i^{\theta-1}\right) &= \sum_{i=1}^{n} \log(\theta x^{\theta-1}) = \sum_{i=1}^{n} \log(\theta) + \sum_{i=1}^{n} \log(x_i^{\theta-1}) \\ &= n\log(\theta) + \sum_{i=1}^{n} (\theta-1)\log(x_i). \end{aligned}$$

Taking the derivative with respect to θ and setting equal to zero we have

$$l'(\theta) = 0 \iff \frac{n}{\theta} + \sum_{i=1}^{n} \log(x_i) = 0 \iff \hat{\theta} = \frac{-n}{\sum_{i=1}^{n} \log(x_i)}.$$

(b) We have

$$L(\theta) = \prod_{i=1}^{n} e^{-(x_i-\theta)},$$

if $x_i \ge \theta$ for all i, otherwise $L(\theta) = 0$. Then

$$l(\theta) = \log[L(\theta)] = \log\left(\prod_{i=1}^{n} e^{-(x_i-\theta)}\right) = \sum_{i=1}^{n} \log(e^{-(x_i-\theta)}) = -\sum_{i=1}^{n}(x_i-\theta).$$

Taking the derivative with respect to θ we have

$$l'(\theta) = -\sum_{i=1}^{n}(-1) = n.$$

Hence $l(\theta)$ is an increasing function. Thus $\hat{\theta} = \min\{X_1, \ldots, X_n\}$. △

6.2.07 Let X have a gamma distribution with $\alpha = 4$ and $\beta = \theta > 0$.
(a) Find the Fisher information $I(\theta)$.

Solution. We have $f(x;\theta) = \frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{x^2}{2\theta}\right)$. The Fisher information is

$$I(\theta) = -E\left[\frac{\partial^2 \log f(x;\theta)}{\partial\theta^2}\right].$$

Then

$$\begin{aligned} \log f(x;\theta) &= -\log(\Gamma(\alpha)) - \alpha\log(\beta) + \log(x^{\alpha-1}) - \frac{x}{\beta} \\ \implies \frac{\partial \log f(x;\theta)}{\partial\theta} &= -\frac{\alpha}{\beta} + \frac{x}{\beta^2} \\ \implies \frac{\partial^2 \log f(x;\theta)}{\partial^2\theta} &= \frac{\alpha}{\beta^2} - \frac{2x}{\beta^3} \\ \implies -E\left[\frac{\partial^2 \log f(x;\theta)}{\partial\theta^2}\right] &= -\frac{\alpha}{\beta^2} + \frac{2E[X]}{\beta^3} = -\frac{\alpha}{\beta^2} + \frac{2\alpha}{\beta^2} = \frac{\alpha}{\beta^2}. \end{aligned}$$

Thus $I(\theta) = \frac{\alpha}{\beta^2} = \frac{4}{\theta^2}$. △

(b) If $X_1, X_2, \ldots, X_n$ is a random sample from this distribution, show that the mle of θ is an efficient estimator of θ.

Solution. First note that the mle $\hat{\theta}$ of θ is $\frac{1}{\alpha}\overline{X}$. Then

$$\mathrm{Var}(\hat{\theta}) = \mathrm{Var}(\frac{1}{\alpha}\overline{X}) = \frac{1}{\alpha^2}\mathrm{Var}(\overline{X}) = \frac{\beta^2}{\alpha} = \frac{\theta^2}{4}.$$

Thus the mle of θ is an efficient estimator of θ since $\mathrm{Var}(\hat{\theta}) = \frac{1}{I(\theta)}$. $\triangle$